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The Chi-Square Test for Hierarchical Dependency (THD) is presented. The THD tests the hypothesis that all members of a population who possess a certain skill are a subset of the members who possess another skill. This hypothesis is basic to the writings of several prominent theorists, such as Gagne and Piaget. The THD is designed as an improvement over existing tests of this hypothesis, e.g., the Test of Inclusion. The model, assumptions, formula derivations, and procedures for the THD are explained. An illustrative example of an application of the THD, involving two spatial skills, is provided. (Author/CTM)

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A CHI-SQUARE TEST FOR HIERARCHICAL DEPENDENCY

A Paper
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Abstract

The Chi-Square Test for Hierarchical Dependency (THD) is presented. The THD tests the hypothesis that all members of a population who possess a certain skill are a subset of the members who possess another skill. This hypothesis is basic to the writings of several prominent theoreists, such as Gagne and Piaget. The THD is designed as an improvement over existing tests of this hypothesis, e.g., the Test of Inclusion. The model, assumptions, formula derivations, and procedures for the THD are explained. An illustrative example of an application of the THD is also provided.

A CHI-SQUARE TEST FOR HIERARCHICAL DEPENDENCY

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The notion of hierarchical dependencies is basic to the writings of several prominent theorists in education and psychology (e.g., Gagné and Piaget). The existence of a hierarchical dependency suggests that mental skills are acquired in an invariant sequence among all people, i.e., all people must be able to display a lower-order prerequisite skill before being able to display a higher-order dependent skill. This notion may be stated by the hypothesis that, within a population, all members who possess a certain skill are a subset of the members who possess another skill.

Typically, research designed to assess the validity of posited hierarchical dependencies have used one of the following statistical procedures: Gagné and Paradise's index of positive transfer, Guttman's coefficient of reproducibility, or the phi-correlation coefficient. The major deficiencies of these statistical procedures for testing hypotheses of hierarchical dependencies are summarized in an article by White (1974). In the same article, White recommends the use of the Test of Inclusion (White and Clark, 1973) to replace existing statistical procedures. A search of the literature revealed that the Test of Inclusion is the "state of the art" in the field at this time.

White and Clark's (1973) Test of Inclusion, although a significant improvement over earlier statistical procedures, also has several major limitations. The procedure requires the measurement of the posited higher-order and lower-order skills for a sample of subjects using two or more

questions for each skill. The hierarchical dependency hypothesis being evaluated by the Test of Inclusion is rejected when the number of people observed who answer all the higher-order skill questions correctly and concurrently answer all the lower-order skill questions incorrectly exceeds a critical value. This critical value is estimated from the observed data assuming the dependency hypothesis under investigation is true. Two major problems with the Test of Inclusion have been observed: (1) the procedure assumes that the probability of answering each question for a particular skill is the same, thus restricting the use of the procedure to studies utilizing one type of question format; and (2) the procedure can yield uninterpretable estimates of probabilities, i.e., negative probability values and probabilities exceeding one.

The Chi-Square Test for Hierarchical Dependency (THD) summarized herein is a statistical procedure designed to eliminate many of the major weaknesses inherent in the Test of Inclusion and other earlier statistical procedures. The THD and the Test of Inclusion differ not only in their conceptual approach, but also in their basic assumptions. For the THD, measures of the posited higher-order and lower-order skills are obtained using one or more questions for each skill. The Test involves the computation of expected values using a model developed to describe the distribution of the population assuming a particular dependency hypothesis. Unknown parameters are estimated and a χ^2 statistic is computed which measures the discrepancy between the data and the dependency hypothesis. The hierarchical dependency hypothesis is rejected when χ^2 exceeds a predetermined value. Details of the THD are presented in the next session. An illustrative example is provided in the last section.

Model/Assumptions

Suppose that a study is designed to investigate whether a hierarchical dependency exists between a skill A and a skill B. Assume that the literature suggests that skill A is a prerequisite to the development of skill B. To test the hypothesis that skill B is hierarchically dependent on skill A, a random sample of N subjects is drawn from the population of interest. The subjects answer j questions measuring skill A and k questions for skill B. These measures may be either selection-or supply-type.

The measures can be represented as

$$X_1, X_2, \dots, X_i, \dots, X_j \quad \text{for skill A,}$$

and

$$Y_1, Y_2, \dots, Y_i, \dots, Y_k \quad \text{for skill B.}$$

The value "1" denotes a correct response and the value "0" an incorrect response.

For the THD it is assumed that (1) if the subjects have skill A, they will be able to answer all the questions for skill A correctly (i.e., $X_i = 1$ for $i = 1, \dots, j$); and (2) if they have skill B, they will answer all the questions for skill B correctly (i.e., $Y_i = 1$ for $i = 1, \dots, k$).

If a subject does not have skill A, the probabilities associated with his/her response to skill A questions is given by

$$P(X_i = 1) = p_{Ai}$$

and

$$P(X_i = 0) = 1 - p_{Ai} = q_{Ai}, \text{ for } i = 1, \dots, j.$$

Similarly, if a subject does not have skill B, then

$$P(Y_i = 1) = p_{Bi}$$

and

$$P(Y_i = 0) = 1 - p_{Bi} = q_{Bi}, \text{ for } i = 1, \dots, k.$$

In addition, if a subject does not have skill A, responses to the skill A questions are assumed to be independent. Similarly for skill B. If both skills are unknown, the X's and Y's are assumed to be independent.

For the THD the proportion of the population having neither skill is represented as π_{00} , π_{A0} is used to represent the proportion having skill A, but not skill B; π_{0B} the proportion having skill B, but not A; and π_{AB} the proportion having both skills. The hypothesis that skill B is hierarchically dependent on skill A can be expressed as

$$H_0: \pi_{0B} = 0.$$

The general alternative hypothesis is

$$H_A: \pi_{0B} > 0.$$

The unknown parameters under H_0 are π_{00} , π_{A0} , and π_{AB} . In order to derive the procedures for the THD, two other unknown parameters are defined: π_x and $\pi_{y|x}$. π_x is the probability that a randomly selected subject has skill A; $\pi_{y|x}$ is the probability that a randomly selected subject has skill B, given that the subject has skill A. The basic relationships among the unknown parameters are

$$\pi_x = \pi_{A0} + \pi_{AB},$$

$$\text{and } \pi_{y|x} = \frac{\pi_{AB}}{\pi_{A0} + \pi_{AB}}.$$

The first step in deriving the procedures for THD is to estimate the unknown parameters π_x and $\pi_{y|x}$. To estimate these parameters, it is assumed that p_{Ai} and p_{Bi} are known. Based on THD assumptions, the indicator variables X(for skill A) and Y(for skill B) can be defined as follows:

$$X = \{1 \text{ if } X_i = 1 \text{ for } i = 1, \dots, j, 0 \text{ otherwise}\},$$

$$Y = \{1 \text{ if } Y_i = 1 \text{ for } i = 1, \dots, k, 0 \text{ otherwise}\}.$$

Note that X and Y take on the value of 1 only if all questions are answered correctly.

Using X and Y to classify the N subjects, the results are denoted as in Table 1.

Table 1
Observed Frequency Distribution

	X		
	0	1	
Y	0	n_{00} n_{10}	n_{+0}
	1	n_{01} n_{11}	n_{+1}
		n_{0+} n_{1+}	$n_{++} = N$

Since the counts n_{00} , n_{01} , n_{10} , and n_{11} are a set of sufficient statistics for the parameters π_{00} , π_{A0} , and π_{AB} , attention can be focused on the analysis of the 2×2 joint frequency distribution presented in Table 1.

The probability that a randomly selected subject answers all the questions for skill A correctly, given that the subject does not have skill A is p_1 . Similarly, p_2 is the probability that a randomly selected subject answers all the questions for skill B correctly, given that the subject does not have skill B. The values of p_1 and p_2 can be obtained as follows:

$$p_1 = \prod_{i=1}^j p_{Ai} \text{ and } p_2 = \prod_{i=1}^k p_{Bi}.$$

Also, let

$$q_1 = 1 - p_1$$

and

$$q_2 = 1 - p_2.$$

Given these definitions and assumptions, a table of expected values is easily calculated. The results are given in Table 2

Table 2
Table of Expected Values Under H_0

Y	X		
	0	1	
0	$E_{00} = (q_1 q_2 - q_1 q_2 \pi_x)N$	$E_{10} = (p_1 q_2 + q_1 q_2 \pi_x - q_2 \pi_x \pi_{y x})N$	$(q_1 - q_2 \pi_x \pi_{y x})N$
	$E_{01} =$	$E_{11} =$	
1	$(q_1 p_2 - q_1 p_2 \pi_x)N$	$(p_1 p_2 + q_1 p_2 \pi_x + q_2 \pi_x \pi_{y x})N$	$(p_2 + q_2 \pi_x \pi_{y x})N$
	$(q_1 - q_1 \pi_x)N$	$(p_1 + q_1 \pi_x)N$	N

The problem of estimating π_x and $\pi_{y|x}$ is complex and the details are beyond the scope of this paper. Efficient estimators of π_x and $\pi_{y|x}$ can be generated using the minimum modified chi-square technique (Rao, 1965).

The estimators are

$$\hat{\pi}_x = 1 - \frac{1}{Nq_1 \left[\frac{q_2^2}{n_{00}} + \frac{p_2^2}{n_{01}} \right]} \quad \text{and}$$

$$\hat{\pi}_{y|x} = \frac{p_1 q_2 + q_1 q_2 \hat{\pi}_x - \frac{n_{10}}{N}}{q_2 \hat{\pi}_x}.$$

Substitution of the estimates of π_x and $\pi_{y|x}$ into the expressions given

in Table 2, gives the following estimated expected values.

$$\hat{E}_{00} = \frac{\frac{q_2^2}{n_{00}} + \frac{p_2^2}{n_{01}}}{\frac{q_2^2}{n_{00}} + \frac{p_2^2}{n_{01}}},$$

$$\hat{E}_{10} = n_{10},$$

$$\hat{E}_{01} = \frac{\frac{q_2^2}{n_{00}} + \frac{p_2^2}{n_{01}}}{\frac{q_2^2}{n_{00}} + \frac{p_2^2}{n_{01}}},$$

and

$$\hat{E}_{11} = n_{11}.$$

These estimated expected values can be compared to the observed frequencies n_{00} , n_{10} , n_{01} , and n_{11} and a χ^2 statistic calculated. For the THD application described herein, the value of χ^2 is

$$\chi^2 = \frac{(\hat{E}_{00} - n_{00})^2}{\hat{E}_{00}} + \frac{(\hat{E}_{01} - n_{01})^2}{\hat{E}_{01}} + \frac{(\hat{E}_{10} - n_{10})^2}{\hat{E}_{10}} + \frac{(\hat{E}_{11} - n_{11})^2}{\hat{E}_{11}}.$$

Simplification gives

$$\chi^2 = \left(\frac{q_2^2}{n_{00}} + \frac{p_2^2}{n_{01}} \right) \left(\frac{n_{00}^2}{q_2^2} + \frac{n_{01}^2}{p_2^2} \right) - n_{0+}.$$

Under the hypothesis that skill B is hierarchically dependent on skill A ($H_0: \pi_{0B} = 0$), the random variable χ^2 has a sampling distribution approximated very closely by a chi-square distribution with one degree of freedom. For the THD, H_0 is rejected when the observed χ^2 value exceeds a predetermined critical value. Large χ^2 values indicate lack of support for the test hypothesis H_0 and/or some or all of the assumptions underlying the THD.

Whether or not H_0 is accepted, the THD should be repeated for the opposite H_0 . In the present context, the opposite hypothesis would be that skill A is hierarchically dependent on skill B. The combined results from the two THD provide maximum information regarding possible hierarchical relationships between the two skills. In studies investigating the hierarchical ordering among three or more skills, the two THD could be repeated for all possible pairs of skills.

Example

Part of a study by Guay (1975) investigated whether a hierarchical dependency relationship exists between two spatial abilities. The 112 subjects in the study were administered two questions for each skill. The questions were all one-right-answer multiple-choice items with three alternatives. The observed-data is presented in Table 3.

Table 3
Observed Joint Frequency Distribution
for Sample Data

		Spatial Skill A		
		Questions Correct	0	1
Spatial Skill B	0	0	12	16
	1	2	30	26
	2	0	6	20

To test the hypothesis that spatial skill B is hierarchically dependent on spatial skill A, or $H_0: \pi_{0B} = 0$, the observed frequencies are collapsed into the 2×2 table shown in Table 4.

Table 4
Observed Joint Frequency Distribution
Under X and Y Conditions
for Sample Data

		Spatial Skill A	
		X	
		0	1
Spatial Skill B	Y		
	0	$n_{00}=44$	$n_{10}=42$
	1	$n_{01}=6$	$n_{11}=20$
		$n_{0+}=50$	$n_{1+}=62$
			$N=112$

Since the probability of guessing the correct answer to each question for spatial skill B is $1/3$, the value of p_2 is $1/9$. The value of q_2 is therefore, $8/9$.

By substituting the values of p_2 , q_2 , n_{00} , n_{01} , and n_{0+} into the simplified χ^2 formula, a χ^2 statistic of .077 is obtained ($p=.78$). An observed χ^2 value of 3.84 or greater would cause rejection of H_0 at the 5% level. Since the observed χ^2 statistic is well below the critical value, H_0 is not rejected.

If the THD is repeated for the hypothesis that spatial skill A is hierarchically dependent on spatial skill B, or $H_0: \pi_{0A} = 0$, a χ^2 statistic of 243.743 is observed ($p < .0001$). In this case the observed χ^2 statistic substantially exceeds the critical value, so H_0 is rejected.

When White and Clark's (1973) Test of Inclusion is applied to this sample data to test the hypothesis that spatial skill B is dependent on spatial skill A, i.e., $H_0: \pi_{0B} = 0$, H_0 is also not rejected at the 5% level. The result is based on the fact that none of the 112 subjects in the study

answered the two skill B questions correctly while concurrently answering the two skill A questions incorrectly (see Table 3). The probability of such an outcome was computed to be .75. Since this value was less than .95, H_0 was retained. For this example, however, the intermediate steps in the calculation of White and Clark's procedure involve estimates of probabilities lying outside the interval $[0,1]$.

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